Finite Element Modeling of Exotic Options

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Abstract

The Finite Element Method is a well-studied and well-understood method of solving partial differential equations. Its applicability to financial models formulated as PDEs is demonstrated. Its advantage concerning the computation of accurate "Greeks" is delineated. This is demonstrated with various exotic options.

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1 Introduction

Many pricing models can be cast into continuous time and as a result will naturally lead to partial differential equations. These types of PDEs are usually linear and parabolic. In order to avoid clutter in notation we restrict our attention to the case of linear models depending on maximal two factors.[1] These models have been solved traditionally with Finite Differences (FD). Many different FD techniques exist ([1], ch. 2); the most important have been introduced to financial problems ([28], ch. 15; [14], ch. 10; [50], ch. 16-22; [15]). The usefulness of Finite Elements (FE) has been recognized by many authors ([24], p. 47; [14], p. 212; [16], p. 1664; [17], p. 582; [18], p. 586; [46]; [9], [55], sec. 2.5.4) but to our knowledge the first to explore this idea in some more detail were [31], [32], [21], [22], [23], and [48].

These authors have shown that FE approaches offer some advantages:

- A solution for the entire domain is computed, instead of isolated nodes as in the case with FD codes.
- The boundary conditions involving derivatives are difficult to handle with FD ([20], p. 501). Neumann conditions, however, are often easier to obtain than Dirichlet conditions when estimating the behaviour of the option when the price of the underlying goes to infinity. FE techniques can incorporate boundary conditions involving derivatives easily.
- In addition, FE can easily deal with high curvature. In most FE codes this is achieved by adaptive remeshing, a technique well-developed in theory and in practice.

In this paper we will concentrate on some further advantages of FE:

- The irregular shapes of the PDE's domain can easily be handled while in a FD setting, the placing of the gridpoints is difficult. These irregular domains arise naturally when knock-out barriers are imposed on a multiple-asset option. Irregular shapes can also arise when only parts of the PDE's domain are to be approximated numerically because some parts can be determined by financial reasoning.
- Most academic papers are concernened with pricing only while most practioners are at least as much interested in measures of sensitivity to those prices. Some of these measures of sensitivity, commonly called Greeks, can be obtained more exactly with FE.
- Many FE codes (such as *PDE*ase*2D*^{TM} used for this paper) allow local refinement. This allows precise local information without having to solve the problem with accuracy on the entire domain. *PDE*ase*2D*^{TM} also employs adaptive remeshing. This feature automatically leads to local refinement in

[1] Most PDE-based option and bond pricing models belong to this class of problem. Notable exceptions are the *nonlinear* models with transaction cost ([37]; [50], ch. 13; [51]) and the 3-factor swaption model by Dempster and Hutton ([12]; [13]). These models can also be solved with FE, but this will not be demonstrated here; see, for instance [54].

areas of high curvature, for example near to the strike price or close to the barrier.

We will demonstrate these ideas with options which are currently traded in the marketplace. Some of them are listed on stock exchanges. While some of these products are a simple application of the FE approach, many are more sophisticated. We present an approach for valueing options on baskets with various barrier features. This implies a two step procedure: First, some PDEs have to be solved in order to get boundary conditions, and second, another PDE has to be approximated numerically to price the product.

2 General Outline

The Pricing PDE Our aim is to explain some features from FE modeling which are especially useful for option pricing. As in most codes available today this takes place within a *hybrid FD/FE*[2] framework. This method discretizes time with FD and the spatial variables with FE, and has been, until today, the predominant way of dealing with time in FE analysis. Technical derivations with increasing levels of rigor can be found in [8], [4], [1], and [45]. We convert the original backward parabolic problem into a forward parabolic problem to be in accordance with most numerical literature. The interpretation of $\tau = T - t$ therefore is *time to maturity* so that the task is to an approximate solution to the following problem:[3]

$$
\begin{aligned}
\frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} &+ \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2} + \qquad (1)\\
(r-q_1)S_1\frac{\partial f}{\partial S_1} + (r-q_2)S_2\frac{\partial f}{\partial S_2} &= rf + \frac{\partial f}{\partial \tau}\\
f(S_1, S_2, 0) &= g_1(S_1, S_2) \text{ in } D \qquad (2)\\
f(S_1, S_2, \tau) &= g_2(S_1, S_2, \tau) \text{ on } R_1 \qquad (3)\\
\frac{\partial f}{\partial n} &= g_3(S_1, S_2, \tau) \text{ on } R_2 \qquad (4)\\
R_1 \cup R_2 &= R \qquad (5)
\end{aligned}
$$

D is the interior of the convex domain, and R constitutes the boundary. $\frac{\partial f}{\partial n}$ denotes the gradient perpendicular to the boundary. Although boundary conditions eq. (3) and eq. (4) and initial conditon eq. (2) may not be compatible, the problem is well-posed [50]. The equations above can be used to price European options of many kinds as the examples in the following chapter will show. We employ a two-asset formulation of the Black-Scholes equation because the extension to more dimensions is fairly straightforward from a financial and a numerical point of view, since this approach incorporates correlations between the assets and allows for Finite Elements with different geometric shapes. A general FE solution for European and American options has been delivered by [23]. The pricing of American options, however, is more

[2]This term stems from Darrell Duffie; the typical name for this approach in the mathematical and engineering literature is *time-dependent FE methods.*

[3]All notation is based on [28] with the only exception of S_n denoting the price of the nth underlying (instead of the price on the underlying S at time n.)

difficult because an early exercise has to be taken into account. In the PDE setting this naturally leads to moving boundary problems [50] which can also be solved with FE [11]. In the option pricing setting there are currently two approaches: Either the nodes in the elements are manipulated in the same way as they are in a FD setting ([22], p. 7f), or the problem is reformulated as a variational inequality which is solved with FE ([50], p. 410ff).

The problem stated above is a special case of the convection-diffusion problem which has been studied for many decades. Therefore, many numerical techniques are available. One of these is the FEM which is outlined in many textbooks; see for instance [4], [8], and [52]. Here, we do not want to add another general outline of an FE procedure for parabolic problems. Instead, we want to highlight some features which are useful for option pricing in a readable manner. For European options as stated in eq. (1) to (5) the hybrid FD/FE method leads to the following system of *ordinary* differential equations.

$$\boldsymbol{q} = \boldsymbol{A}\,\boldsymbol{u} + \boldsymbol{B}\,\dot{\boldsymbol{u}} \tag{6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_{\circ} \tag{7}$$

Thus the problem of solving a PDE has been reduced to solving a system of ordinary differential equations. This initial value problem is usually solved with a FD technique. For a discussion of the most appropriate ones in this setting compare [4] or [8]. The assembly of the elements has been performed implicitly.

Adaptive Time Steps Our software uses Crank-Nicholson time differencing to solve the system above. In order to get an estimate of the error incurred by the time steps a three-step approach is used. First, a half-step solution estimates the values at the mid-step; then, a full-step estimates the values at step end. Then a half-step advances from mid-step to end-step. These two estimates of the end-step value allow the determination of a cubic time term in a Taylor expansion of the solution in time. This cubic term in under the control of the user (by the command `errlim`).

Adaptive Meshing Since *PDE*ase*2D*^{TM} controls adaptively timesteps and spatial gridding there is a problem of dividing the errors between temporal and spatial controls. The technique employed here is propriatory. So, we will concentrate on the spatial meshing. The software uses triangular elements.[4] This allows to discretrize any domain with piecewise linear boundaries. Curved boundaries can only be dicretized approximately but this is no disadvantage for financial applications where all the boundaries are linear. In areas of high curvature the triangular elements are divided into two new triangular elements. This process is repeated until some error limit is met.

The Greeks Besides option premiums, one is also interested in the *Greeks*. The FEM is especially for *Delta* ($\Delta = \frac{\partial f}{\partial S_i}$) and *Gamma* ($\Gamma = \frac{\partial^2 f}{\partial S_j \partial S_i}$), well-suited because

[4] *PDE*ase*2D*^{TM} treats problems with only one spatial variable as having two spatial variables. The second variable is a dummy.

it delivers a polynomial approximation in the spatial variables.[5] The derivatives of polynomials can be easily computed *analytically* and as a result very fast.[6] Obviously, for this to work, the shape functions have to be at least quadratic. For higher Greeks, like *Speed*, ($\frac{\partial^3 f}{\partial S_i \partial S_j \partial S_k}$; compare ([55], p. 78) this approach becomes complicated due to the fact that many types of elements become admissible. One can improve this procedure by taking the Greeks at the so-called *Moan Points*. Moan points are points of the FE approximation which have exact derivatives ([42], [4], [8]). Since in financial problems, one is usually interested only in solutions to one or several points in the parameter space, the elements can always be constructed in a way that these points of interest become Moan points. Another possible approach is to use low-order shape functions and employ a device called *local smoothing* from the engineering literature [27].

3 Examples

3.1 Barrier Options

3.1.1 Double Barrier

We consider an up-and-out-down-and-out call option continuously monitored,[7] with the following data:

Parameter	Value
Strike price	100
Down-and-out barrier	75
Up-and-out barrier	130
Rebates	none
Interest rate	0.1
Volatility	0.2
Maturity	1 year

Table 1: Data Double Barrier Option

This leads to the following well-posed backward parabolic PDE problem:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \quad (8)$$

$$f(T,S) = \max(S-X,0) \quad (9)$$

$$f(t,75) = 0 \quad (10)$$

$$f(t,130) = 0 \quad (11)$$

The pricing PDE eq. (8) is the famous Black-Scholes equation [5]. Eq. (9) constitutes the payoff at maturity. Eq. (10) and (11) are the knock-out barriers. The analytical

[5] There are pure FE approaches which apply FE in time, too; compare [34], [19]. This, however, is not the general methodology.

[6] The package used for this paper [38] only allows numerical derivatives.

[7] Solutions to problems with discrete monitoring can be found by applying the adjustment formulae by [7] to the continuous-monitoring solution.

solution involves a series which goes from $-\infty$ to ∞ ([26], p. 73). For numerical purposes this series has to be cut off after some finite number of terms. It has been shown in [35] that it is sufficient to consider only the terms from -2 to 2 because all other terms are very close to zero. Here, for the analytical solution, we have taken the terms from -5 to 5.[8] The *root mean sqare error* RMS is controlled by the user. The default, which is used for all other runs, is 0.001 ([38], p. 104). The value of the underlying is varied in order to catch different degrees of the moneyness. Since the program is adaptive in time and space the number of cycles, nodes, and cells are chosen during the solution process by the program. The code to this and all the other problems from this paper can be found on the disk which comes with this paper.[9]

Underlying	Fair Value						
	Analytical	Numerical					
		RMS 0.01		RMS 0.001		RMS 0.0001	
			Error		Error		Error
76	0.27306	0.27376	0.26 %	0.27317	0.04 %	0.27317	0.04 %
80	1.22027	1.22357	0.27 %	1.22092	0.05 %	1.22087	0.05 %
90	2.90287	2.90875	0.20 %	2.90378	0.03 %	2.90378	0.03 %
100	3.52511	3.52456	0.02 %	3.52395	0.03 %	3.52533	0.01 %
110	2.89967	2.89187	0.27 %	2.89670	0.10 %	2.89932	0.01 %
120	1.47489	1.46833	0.44 %	1.47269	0.15 %	1.47458	0.02 %
129	0.13192	0.13137	0.42 %	0.13181	0.08 %	0.13192	0.01 %
Data of FE-Run							
Cycles		25		57		72	
Nodes		223		219		219	
Cells		74		72		130	

Table 2: Results Double Barrier Option

The *root mean sqare error* RMS is controlled by the user. Error is defined as relative deviation:

$$\text{error} = \left| \frac{\text{result} - \text{result}_{\text{FE}}}{\text{result}} \right| \tag{12}$$

The reported errors and differences here and in following sections are based on more significant digits than are shown in the tables.

3.1.2 Single Barrier

The following example is based on the example in ([3], p. 225f).

[8] It is the normal case that analytical solutions to option pricing problems involve infinite series and/or indefinite integrals. This has led ([50], p. 261) to the recommendation *not* to look for analytical solutions (which are usually not easy to find provided they exist; compare ([43], sec. 2.3)) but to solve the PDE with numerical methods directly.

[9] Available from the author upon request.

Parameter	Value
Strike price	100
Up-and-out barrier	110
Rebate	10
Interest rate	0.05
Volatility	0.2
Maturity	0.5 year

Table 3: Data Single Barrier Option

This leads to the following well-posed backward parabolic PDE problem:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{13}$$

$$f(T, S) = \max(S - X, 0) \tag{14}$$

$$f(t, 0) = 0 \tag{15}$$

$$f(t, 110) = 10 \tag{16}$$

Eq. (15) can be interpreted as a knock-out barrier: Once the price of the underlying equity hits zero the company is bankrupt and will not recover in value [33]. Consequently, any call on this equity will be worthless. In eq. (16) a lump sum rebate is introduced.

Und.	Method	Fair value	Delta	Gamma
80	Analytical	0.43223	0.08507	0.01295
	Numerical	0.43221	0.08507	0.01298
	Error	0.0040 %	0.0000 %	0.1965 %
90	Analytical	2.10253	0.26128	0.01999
	Numerical	2.10252	0.26130	0.01992
	Error	0.0003 %	0.0068 %	0.3707 %
100	Analytical	5.60968	0.42205	0.00939
	Numerical	5.60975	0.42204	0.00927
	Error	0.0012 %	0.0014 %	1.3159 %
105	Analytical	7.79972	0.44635	0.00031
	Numerical	7.79971	0.44635	0.00030
	Error	0.0001 %	0.0000 %	3.3333 %
109	Analytical	9.56930	0.43406	-0.00625
	Numerical	9.56929	0.43405	-0.00620
	Error	0.0001 %	0.0029 %	0.8342 %

Table 4: Results Single Barrier Option

3.1.3 Time-dependent Rebates

We consider the same problem as in sec. (3.1.2) except that the rebate becomes a step-like function of time. Each month it doubles, starting with 1. In mathematical terms, eq. (16) has to be replaced by:

$$f(t,110) = \begin{cases} 1, & 0 < t < \frac{1}{12} \\ 2, & \frac{1}{12} < t < \frac{2}{12} \\ 4, & \frac{2}{12} < t < \frac{3}{12} \\ 8, & \frac{3}{12} < t < \frac{4}{12} \\ 16, & \frac{4}{12} < t < \frac{5}{12} \\ 32, & \frac{5}{12} < t < \frac{6}{12} \end{cases} \tag{17}$$

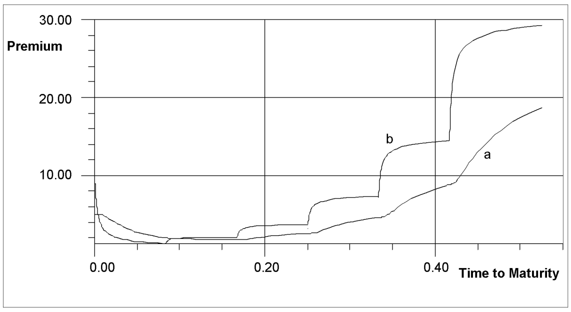

Figure 1: Price of Option as a Function of Time with Underlying at 105 (a) and 109 (b)

Underlying	Fair value	Delta	Gamma
80	0.18569	0.03352	0.00493
90	0.91468	0.13750	0.01774
100	4.48090	0.71799	0.09950
105	9.35280	1.21336	0.08385
109	14.63519	1.37108	-0.00449

Table 5: Results Single Barrier Option with Time-dependent Rebate

3.1.4 Time-dependent Volatilities

One of the most often criticized weaknesses of the Black-Scholes model is its assumption of constant volatility. This assumption, however, can be dropped without leaving the Black-Scholes environment of lognormal returns. One approach is to assume a term structure of volatility. The most simple model for a term structure of volatility is to assume that the volatility is a linear function of time to maturity:

$$\sigma(\tau) = a\,\tau + b \tag{18}$$

No analytical formula is known for volatility models depending on the moneyness and/or time-to-maturity. This leads to volatility surfaces which are widely used ([51], ch. 14.6). The FEM allows one to integrate complicated deterministic volatility models as shown by [32]. Here we will contrast our results to the results reported by [6] using a trinominal tree. Unfortunately, no details on the trinominal tree calculations are provided.

Parameter	Value
Asset price	95
Strike price	100
Down-and-out barrier	90
Interest rate	0.1
Maturity	1 year

Table 6: Data Single Barrier Option with with Time-dependent Volatility

Here we consider constant, increasing, and decreasing volatility:

Problem	Initial volatility	Ending volatility	a	b
1	0.25	0.25	0	0.25
2	0.177	0.306	-0.129	0.306
3	0.306	0.177	0.129	0.177

Table 7: Data Volatility Curve

Unfortunately, [6] does not provide any details on his trinominal tree computations.

Problem	Method	Fair value	Delta
1	Analytical	5.9968	1.120
	FE	5.9969	1.119
	Difference	0.0017 %	0.0894 %
2	Trinominal tree	6.4556	1.146
	FE	6.4632	1.145
	Difference	0.1176 %	0.0873 %
3	Trinominal tree	5.7286	1.093
	FE	5.7169	1.089
	Difference	0.2047 %	0.3673 %

Table 8: Results Volatility Curve

3.2 Power Options

3.2.1 Plain Vanilla Power Option

Power options can be subdivided into symmetric and asymmetric power options according to their payoffs:

- symmetric power call: $\max((S - X)^p, 0)$
- asymmetric power call: $\max(S^p - X, 0)$

The payoffs of the puts can be constructed accordingly. To both types, analytical solutions are available ([55], ch. 30). Here, we will contrast our numerical solution to the analytical solutions for the premium, *Delta* (Δ), and *Gamma* (Γ). As a basis, we take an example from ([55], p. 589) with the following data:

Parameter	Value
Asset price	555
Strike price	550
Interest rate	0.06
Volatility	0.15
Dividend Yield	0.04
Maturity	0.5 year

Table 9: Data Aysmmetric Power Option

In mathematical terms, this can be formulated as following:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \quad (19)$$

$$f(T, S) = \max(S^p - X, 0) \quad (20)$$

$$f(t, 0) = 0 \quad (21)$$

$$f(t, 1000) = S^p - Xe^{-rt} \quad (22)$$

Eq. (22) denotes the value of the option deep in the money. It is common practice to cut off the semi-infinite domain at some point to get a finite domain since most numerical routines apply to finite domains ([39], p. 283; [36]; [49]) although numerical techniques for semi-infinite techniques exist ([25]; [53]). The power parameter p is varied.

p	0.96	0.97	0.98	0.99	1.00
Analytical	0.17614	1.01010	4.08800	12.21638	28.29032
FE	0.17615	1.01080	4.08802	12.21638	28.29040
Difference	0.0037 %	0.0027 %	0.0003 %	0.0001 %	0.0003 %
Δ	0.00892	0.04218	0.13766	0.32420	0.58026
Δ_{FE}	0.00892	0.04219	0.13767	0.32421	0.58026
Difference	0.0215 %	0.0114 %	0.0064 %	0.0016 %	0.0001 %
Γ	0.00038	0.00141	0.00346	0.00570	0.00648
Γ_{FE}	0.00039	0.00145	0.00354	0.00576	0.00647
Difference	3.6430 %	2.7837 %	2.1628 %	1.0376 %	0.0000 %

Table 10: Results Asymmetric Power Calls (Part 1)

p	1.01	1.02	1.03	1.04	1.05
Analytical	53.39500	86.29781	124.81669	167.30009	213.01648
FE	53.39502	86.29778	124.81670	167.30010	213.01650
Difference	0.0000 %	0.0000 %	0.0000 %	0.0000 %	0.0000 %
Δ	0.83817	1.04341	1.19100	1.30579	1.41124
Δ_{FE}	0.83817	1.04340	1.19099	1.30579	1.41124
Difference	0.0004 %	0.0011 %	0.0004 %	0.0003 %	0.0001 %
Γ	0.00516	0.00297	0.00129	0.00048	0.00022
Γ_{FE}	0.00515	0.00291	0.00126	0.00048	0.00022
Difference	0.3056 %	2.1197 %	2.2855 %	0.2417 %	1.0919 %

Table 11: Results Asymmetric Power Calls (Part 2)

3.2.2 Capped Power Option

The Asymmetric Case As mentioned previously, there are closed-form solutions to symmetric and asymmetric power options. But within the market place, only *capped* power calls and puts with a floor are traded,[10] for which an analitical solution is not known. In mathematical terms, the problem is to find a solution to the following PDE with the data from the example in sec. 3.2.1:

$$\begin{aligned}
\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} &= rf & (23)\\
f(T,S) &= \min(\max(S^p - X, 0), C) & (24)\\
f(t,0) &= 0 & (25)\\
f(t,1000) &= 50 & (26)\\
C &= 50 & (27)
\end{aligned}$$

[10] An Example: In Germany, Bankers Trust has issued capped symmetric FX power options on US $ (WKN 822512 - WKN 822521), Swiss Francs (WKN 822374, WKN 822376), and Japanese Yen (WKN 826053f) with a power parameter of $p = 2$.

p	0.96	0.97	0.98	0.99	1.00
Monte Carlo	0.163	0.909	3.442	9.327	18.887
FE	0.165	1.008	3.434	9.332	18.886
Δ_{FE}	0.00814	0.04210	0.10882	0.21931	0.30711
Γ_{FE}	0.00033	0.00141	0.00242	0.00278	0.00097
p	1.01	1.02	1.03	1.04	1.05
Monte Carlo	29.897	39.098	44.745	47.327	48.219
FE	29.893	39.084	44.736	47.326	48.224
Δ_{FE}	0.30619	0.22105	0.11743	0.04658	0.01398
Γ_{FE}	-0.00194	-0.00356	-0.00304	-0.00167	-0.00064

Table 12: Results Capped Asymmetric Power Calls

The Monte Carlo results have been achieved in the most simple way with 1,000,000 samplings.

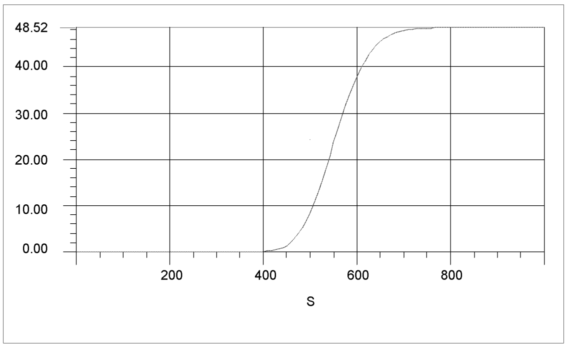

Figure 2: Premium of a Capped Symmetric Power Option

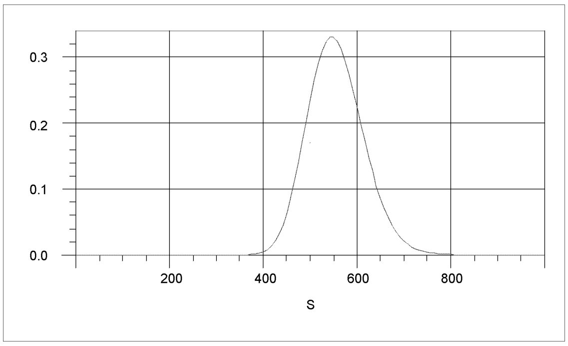

Figure 3: *Delta* Δ of a Capped Symmetric Power Option

The Symmetric Case The more popular capped symmetric power option can be formulated accordingly:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \quad = \quad rf \tag{28}$$

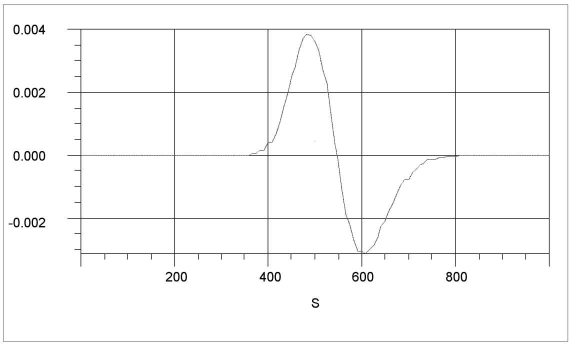

Figure 4: *Gamma* Γ of a Capped Symmetric Power Option

$$
\begin{aligned}
f(T,S) &= \min\left(\max\left((S-X)^p,0\right),C\right) && (29)\\
f(t,0) &= 0 && (30)\\
f(t,1000) &= 50 && (31)\\
C &= 50 && (32)
\end{aligned}
$$

We take the data from table (9) and vary the asset price from out-of-the-money to in-the-money. The power parameter is set to $p = 2$ which is predominant in the market place. We compare our results with a simple Monte Carlo approach based on 1,000,000 samplings.

p	500	550	555	560	600
Monte Carlo	8.47390	23.50052	25.15097	26.78109	37.98719
FE	8.46219	23.51419	25.16434	26.79323	37.97783
Difference	0.1297 %	0.0582 %	0.0532 %	0.1745 %	0.0246 %
Δ_{FE}	0.23545	0.33162	0.32844	0.32311	0.22415
Γ_{FE}	0.00381	-0.00015	-0.00064	-0.00107	-0.00306

Table 13: Results Capped Symmetric Power Calls

3.3 Basket Options

3.3.1 Put on a Basket

For options on baskets, at present there is no known analytical solution ([29], p. 161). Therefore, this option has to be priced with a numerical device or an approximation like ([30]; [41]; [55], ch. 27). The basic idea of these approximations is to combine the volatilities of the underlying and their correlations to a single volatility of the basket. This basket is then treated as a single underlying. Using this approach, the problem of pricing an option on a basket is reduced to pricing an option on a single equity. Accordingly, the models to price options with exotic features can also be applied to options on baskets. Precise error estimates are generally not provided ([29], p. 163). Here, however, we price options on baskets using a multi-dimensional PDE. For a plain vanilla put we first derive the boundary conditions. As one or both underlyings become worth much more than the strike, the price of the options goes to zero. As the price of first underlying is zero, while the second is positive, the value of the option behaves like the value of a plain vanilla put on a single equity. Therefore, the boundary conditions at $S_1 = 0$ and $S_2 = 0$ are the (time-dependent) solution to the basic Black-Scholes problem of pricing a put ([29], p. 162) with strikes at $\frac{X}{w_2}$ and $\frac{X}{w_1}$, respectively. Together with the data this becomes the following PDE problem.

Parameter	Value
First asset price	18
Weight first asset	1
Second asset price	20
Weight second asset	1
Correlation	0.5
Strike price	40
Interest rate	0.1
Dividend Yields	0.0

Table 14: Data Put on a Basket

$$\begin{aligned}
\frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} &+ \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2} + && (33)\\
(r-q_1)S_1\frac{\partial f}{\partial S_1} + (r-q_2)S_2\frac{\partial f}{\partial S_2} &= rf - \frac{\partial f}{\partial t} &&\\
f(S_1, S_2, T) &= \max(0, X - (w_1 S_1 + w_2 S_2)) \text{ in } D && (34)\\
f(S_1, 0, t) &= g(S_1, \frac{X}{w_2}, t) && (35)\\
f(0, S_2, t) &= g(S_2, \frac{X}{w_1}, t) && (36)\\
f(100, S_2, t) &= 0 && (37)\\
f(S_1, 100, t) &= 0 && (38)
\end{aligned}$$

Here, the g functions denote a plain vanilla European put with strikes of $\frac{X}{w_2}$ and $\frac{X}{w_1}$ and appropriate volatilities. We compute the cumulative normal distributions in equations (35) and (36) with an approximation which has four digit accuracy from ([28], p. 243).[11] To compare the results, we also price the put on a basket using a two-dimension binominal tree as implemented by ([26], ch. 3.3). This tree can be interpreted as a simple explicit finite difference scheme; compare ([50], p. 279).

[11] For higher accuracy see also ([28], p. 243f).

Volatility σ_1^2	σ_2^2	Time to Maturity 0.05	0.5	0.95	Premium
0.1	0.1	1.8025	0.9543	0.6043	Tree
		1.8065	0.9543	0.6035	FEM
		0.2204 %	0.0022 %	0.1352 %	Diff.
	0.2	1.8333	1.4756	1.2408	Tree
		1.8341	1.4764	1.2405	FEM
		0.0473 %	0.0512 %	0.0305 %	Diff.
	0.3	1.9118	2.0186	1.9265	Tree
		1.9138	2.0187	1.9270	FEM
		0.1034 %	0.0041 %	0.0242 %	Diff.
0.2	0.1	1.8271	1.4120	1.1607	Tree
		1.8275	1.4127	1.1601	FEM
		0.0236 %	0.0492 %	0.0489 %	Diff.
	0.2	1.8859	1.8835	1.7758	Tree
		1.8856	1.8833	1.7754	FEM
		0.0076 %	0.0125 %	0.0202 %	Diff.
	0.3	1.9816	2.3941	2.4389	Tree
		1.9830	2.3942	2.4389	FEM
		0.0602 %	0.0024 %	0.0004 %	Diff.
0.3	0.1	1.8906	1.8941	1.7649	Tree
		1.8915	1.8948	1.7647	FEM
		0.0451 %	0.0395 %	0.0108 %	Diff.
	0.2	1.9683	2.3301	2.3557	Tree
		1.9687	2.3298	2.3555	FEM
		0.0210 %	0.0138 %	0.0095 %	Diff.
	0.3	2.0739	2.8112	2.9985	Tree
		2.0747	2.8119	2.9979	FEM
		0.0360 %	0.0021 %	0.0181 %	Diff.

Table 15: Results Put Option on a Basket Computed on a Square Domain

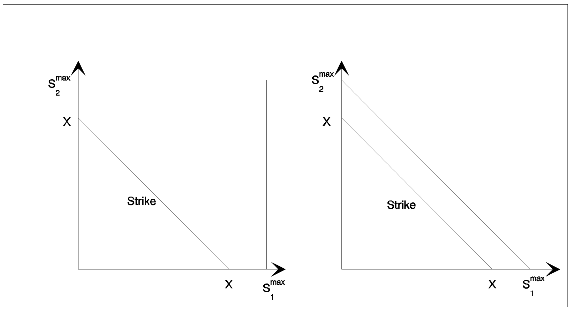

Figure 5: Quadratic and triangular domains for options on baskets

As an alternative to pricing this option on a square domain, we also price it on a triangle, compare fig. (5). Basically, we cut off the section of the domain where the option is totally out of the money and therefore worthless. This, of course, reduces computing time. The boundary conditions (35) to (38) are replaced by:

$$f(S_1, 0, t) = g(S_1, \frac{X}{w_2}, t) \tag{39}$$

$$f(0, S_2, t) = g(S_2, \frac{X}{w_1}, t) \tag{40}$$

$$f(S_1, S_2, t) = 0 \text{ on } \overline{S_1^{\max} S_2^{\max}} \tag{41}$$

Volatility σ_1^2	Volatility σ_2^2	Time to Maturity 0.05	Time to Maturity 0.5	Time to Maturity 0.95	Premium
0.1	0.1	1.8026	0.9543	0.6043	Tree
		1.8035	0.9545	0.6035	FEM
		0.0498 %	0.0155 %	0.0938 %	Diff.
	0.2	1.8333	1.4756	1.2408	Tree
		1.8334	1.4770	1.2407	FEM
		0.0048 %	0.0899 %	0.0097 %	Diff.
	0.3	1.9118	2.0186	1.9265	Tree
		1.9135	2.0187	1.9262	FEM
		0.0919 %	0.0053 %	0.0174 %	Diff.
0.2	0.1	1.8271	1.4120	1.1607	Tree
		1.8265	1.4119	1.1604	FEM
		0.0324 %	0.0095 %	0.0247 %	Diff.
	0.2	1.8859	1.8835	1.7758	Tree
		1.8850	1.8834	1.7753	FEM
		0.0468 %	0.0033 %	0.0256 %	Diff.
	0.3	1.9818	2.3941	2.4389	Tree
		1.9826	2.3940	2.4387	FEM
		0.0426 %	0.0044 %	0.0098 %	Diff.
0.3	0.1	1.8906	1.8941	1.7649	Tree
		1.8908	1.8937	1.7644	FEM
		0.0076 %	0.0218 %	0.0283 %	Diff.
	0.2	1.9683	2.3301	2.3557	Tree
		1.9679	2.3299	2.3555	FEM
		0.0165 %	0.0076 %	0.0081 %	Diff.
	0.3	2.0739	2.8120	2.9985	Tree
		2.0747	2.8120	2.9988	FEM
		0.0368 %	0.0006 %	0.0107 %	Diff.

Table 16: Results Put Option on a Basket Computed on a Triangular Domain

Although it is possible to adjust FD schemes for non-rectangular domains ([40], ch. 2; [2], p. 258f; [1], sec. 1.9; [10]; [43], sec. 3.4) FE are the natural choice. This is even more true for Neumann conditions which are difficult to integrate into more advanced FD schemes in case of curved boundaries. In a FE setting, however, Neumann conditions are even easier to consider than Dirichlet conditions.

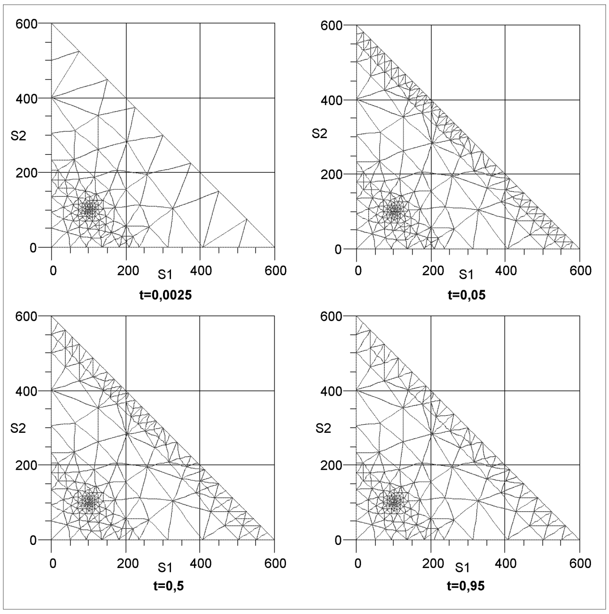

Figure 6: History of FE Mesh

3.3.2 Call on a Basket

In [47] Rubinstein reports results on pricing a call on a basket with the following data using a two-dimensional binominal tree:

Parameter	Value
First asset price	100
Weight first asset	1
Second asset price	100
Weight second asset	1
Strike price	200
Correlation	0.5
Interest rate	0.0953102
Dividend yield first asset	0.0487902
Dividend yield second asset	0.0

Table 17: Data Call on a Basket

With a hundred time steps he achieves the following results:

Volatility		Time to Maturity		
σ_1^2	σ_2^2	0.05	0.5	0.95
0.1	0.1	1.92	8.97	14.70
	0.2	2.72	11.22	17.45
	0.3	3.58	13.70	20.59
0.2	0.1	2.72	11.15	17.28
	0.2	3.45	13.33	20.13
	0.3	4.24	15.72	23.25
0.3	0.1	3.57	13.56	20.25
	0.2	4.24	15.65	23.08
	0.3	4.99	17.94	26.16

Table 18: Rubinstein's example with 100 time steps

In order to achieve higher accuracy we redo the example with 200 time steps. These computations have been performed with an implementation of the two-dimension binominal tree by [26]. We compare these results to FE results using the Black-Scholes formula for calls on $S_1 = 0$ and $S_2 = 0$ as boundary conditions. We solve the appropriate PDE on a triangular domain assuming a 90 degree outward pointing gradient on the third side.

Volatility		Time to Maturity			Method
σ_1^2	σ_2^2	0.05	0.5	0.95	
0.1	0.1	1.9202	8.9685	14.6980	Tree
		1.9198	8.9708	14.6979	FEM
		0.02229 %	0.02548 %	0.00055 %	Difference
	0.2	2.7244	11.2200	17.4460	Tree
		2.7316	11.2200	17.4451	FEM
		0.2643 %	0.0007 %	0.0050 %	Difference
	0.3	3.5738	13.6926	20.5925	Tree
		3.5805	13.6928	20.5872	FEM
		0.1857 %	0.0014 %	0.0258 %	Difference
0.2	0.1	2.7222	11.1506	17.2815	Tree
		2.7317	11.1497	17.2804	FEM
		0.3368 %	0.0087 %	0.0061 %	Difference
	0.2	3.4482	13.3308	20.1281	Tree
		3.4563	13.3295	20.1273	FEM
		0.2357 %	0.0093 %	0.0038 %	Difference
	0.3	4.2420	15.7150	23.2494	Tree
		4.2473	15.7138	23.2472	FEM
		0.1252 %	0.0079 %	0.0097 %	Difference
0.3	0.1	3.5689	13.5534	20.2429	Tree
		3.5779	13.5501	20.2406	FEM
		0.2530 %	0.0244 %	0.0110 %	Difference
	0.2	4.2398	15.6421	23.0708	Tree
		4.2453	15.6408	23.0686	FEM
		0.1310 %	0.0090 %	0.0095 %	Difference
	0.3	4.9846	17.9382	26.1506	Tree
		4.9878	17.9362	26.1459	FEM
		0.0640 %	0.0110 %	0.01798 %	Difference

Table 19: Rubinstein's example with Dirichlet Boundary Conditions

Redoing the above example assuming a zero gradient on $S_1 = 0$ and $S_2 = 0$ leads to slightly less accurate results. This Neumann condition is obviously the easiest to apply.

Volatility		Time to Maturity			Method
σ_1^2	σ_2^2	0.05	0.5	0.95	
0.1	0.1	1.9202	8.9685	14.6980	Tree
		1.9198	8.9712	14.6982	FEM
		0.0223 %	0.0303 %	0.0020 %	Difference
	0.2	2.7244	11.2199	17.4460	Tree
		2.7316	11.2210	17.4461	FEM
		0.2644 %	0.0097 %	0.0004 %	Difference
	0.3	3.5738	13.6926	20.5925	Tree
		3.5805	13.6942	20.5889	FEM
		0.1855 %	0.0112 %	0.0177 %	Difference
0.2	0.1	2.7222	11.1507	17.2815	Tree
		2.7314	11.1506	17.2814	FEM
		0.3368 %	0.0005 %	0.0006 %	Difference
	0.2	3.4482	13.3308	20.1281	Tree
		3.4563	13.3306	20.1286	FEM
		0.2358 %	0.0013 %	0.0028 %	Difference
	0.3	4.2420	15.7150	23.2494	Tree
		4.2473	15.7152	23.2488	FEM
		0.1253 %	0.0016 %	0.0024 %	Difference
0.3	0.1	3.5689	13.5534	20.2429	Tree
		3.5779	13.5517	20.2421	FEM
		0.2530 %	0.0129 %	0.0035 %	Difference
	0.2	4.2398	15.6421	23.0708	Tree
		4.2454	15.6420	23.0704	FEM
		0.1313 %	0.0006 %	0.0017 %	Difference
	0.3	4.9846	17.9382	26.1506	Tree
		4.9878	17.9376	26.1477	FEM
		0.0640 %	0.0031 %	0.0109 %	Difference

Table 20: Rubinstein's example with Neumann Boundary Conditions

3.3.3 Single Barrier Knock-Out Call on a Basket

Without Rebate

For the knock-out call on a basket the boundaries for $S_1 = 0$ and $S_2 = 0$ first have to be computed numerically due to the reasons explained in chapter 3.1. The third

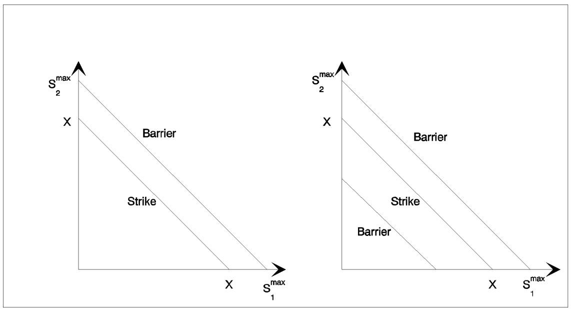

Figure 7: Domains of a Single and Double Barrier Knock-out Call

boundary is the rebate $R = 10$. This leads to the following system of PDEs. Eq. (42) to (45) denote a barrier call on S_1 with $S_2 = 0$. Accordingly, eq. (46) to (49) denote a barrier call on S_2 with $S_1 = 0$

$$
\begin{aligned}
\frac{\partial f_1}{\partial t} + rS\frac{\partial f_1}{\partial S_1} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f_1}{\partial S_1^2} &= rf_1 && (42)\\
f(T, S_1) &= \max(S_1 - X, 0) && (43)\\
f(t, 0) &= 0 && (44)\\
f(t, S_1^{\max}) &= R && (45)\\
\frac{\partial f_2}{\partial t} + rS\frac{\partial f_2}{\partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f_2}{\partial S_2^2} &= rf_2 && (46)\\
f(T, S_2) &= \max(S_2 - X, 0) && (47)\\
f(t, 0) &= 0 && (48)\\
f(t, S_2^{\max}) &= R && (49)\\
\frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f_3}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f_3}{\partial S_2^2} &+ \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 f_3}{\partial S_1 \partial S_2} + && (50)\\
(r - q_1)S_1\frac{\partial f_3}{\partial S_1} + (r - q_2)S_2\frac{\partial f_3}{\partial S_2} &= rf_3 - \frac{\partial f_3}{\partial t} &&\\
f_3(S_1, S_2, T) &= \max(0, X - (w_1 S_1 + w_2 S_2)) \text{ in } D && (51)\\
f_3(S_1, 0, t) &= f_1(S_1, t) && (52)\\
f_3(0, S_2, t) &= f_2(S_2, t) && (53)\\
f_3(S_1, S_2, t) &= R \text{ on } \overline{S_1^{\max} S_2^{\max}} && (54)
\end{aligned}
$$

The parameters are again taken from Rubinstein's example. We solve this problem as a system although it could be solved sequentially. Solving this problem as system avoids having to feed the numerical solutions back into the program. We do not have a direct way of checking the results but the premiums should be below the ones from Rubinstein's example due to the knock-out feature (which they are).

Volatility		Time to Maturity		
σ_1^2	σ_2^2	0.05	0.5	0.95
0.1	0.1	1.7416	0.4645	0.1771
	0.2	1.5198	0.1532	0.0581
	0.3	1.0738	0.0643	0.0246
0.2	0.1	1.5218	0.1568	0.0608
	0.2	1.1074	0.0727	0.0273
	0.3	0.7361	0.0375	0.0140
0.3	0.1	1.0199	0.0665	0.0264
	0.2	0.7368	0.0381	0.0144
	0.3	0.5108	0.0225	0.0083

Table 21: Results Knock-out Call on a Basket without Rebate

With Rebate By introducing a rebate of $R = 10$ the problem above looses a lot of its curvature. Again, we do not have a direct way of checking the results. By arbitrage considerations, however, each premium should be worth more than without rebate and less than in Rubinstein's example. This is satisfied as can be checked easily by inspecting the tables (21) and (19).

Volatility		Time to Maturity		
σ_1^2	σ_2^2	0.05	0.5	0.95
0.1	0.1	1.9161	6.5008	7.8353
	0.2	2.6963	7.0624	8.0269
	0.3	3.4802	7.5058	8.2390
0.2	0.1	2.6944	7.0598	8.0308
	0.2	3.3708	7.4913	8.2597
	0.3	4.0531	7.8185	8.4395
0.3	0.1	3.4771	7.5017	8.2416
	0.2	4.0517	7.8177	8.4410
	0.3	4.6210	8.0668	8.5942

Table 22: Results Knock-out Call on a Basket with Rebate

3.3.4 Double Barrier Knock-Out Call on a Basket

In addition to the example above, in table (24) we introduce a second down-and-out barrier at a value of the basket of 100. No rebate is paid on this barrier. The domain now turns into an irregular strip.

Volatility		Time to Maturity		
σ_1^2	σ_2^2	0.05	0.5	0.95
0.1	0.1	1.9272	6.4709	7.8227
	0.2	2.6108	7.0515	8.0222
	0.3	3.4162	7.5003	8.2368
0.2	0.1	2.6218	7.0465	8.0245
	0.2	3.3089	7.4847	8.2566
	0.3	4.0078	7.8149	8.4379
0.3	0.1	3.4221	7.4949	8.2383
	0.2	4.0064	7.8139	8.4394
	0.3	4.5877	8.0653	8.5907

Table 23: Results Double Barrier Knock-out Call on a Basket

Volatility		Time to Maturity		
σ_1^2	σ_2^2	0.05	0.5	0.95
0.1	0.1	1.9272	6.4709	7.8227
	0.2	2.6108	7.0515	8.0222
	0.3	3.4162	7.5003	8.2368
0.2	0.1	2.6218	7.0465	8.0245
	0.2	3.3089	7.4847	8.2566
	0.3	4.0078	7.8149	8.4379
0.3	0.1	3.4221	7.4949	8.2383
	0.2	4.0064	7.8139	8.4394
	0.3	4.5877	8.0653	8.5907

Table 24: Results Knock-out Call on a Basket with Rebate

The plausibility of these results can be checked with table (24). They are slightly lower than in the example without down-and-out barrier. Since this additional barrier is deeply out of the money it does have only little impact.

3.3.5 Capped Call on a Basket

Analytical Boundary Conditions The pricing of this product[12] leads to the PDE eq. (34) with the following initial and boundary conditions:

$$f(S_1, S_2, 0) = \min(\text{cap}, \max(0, X - (w_1 S_1 + w_2 S_2))) \quad (55)$$

$$f(S_1, 0, t) = g(S_1, \frac{X}{w_2}, t) - g(S_1, \text{cap}, t) \quad (56)$$

$$f(0, S_2, t) = g(S_2, \frac{X}{w_1}, t) - g(S_2, \text{cap}, t) \quad (57)$$

$$f(S_1, S_2, t) = 0 \text{ on } \overline{S_1^{\max} S_2^{\max}} \quad (58)$$

[12] In Germany, examples for capped basket options are WKN 822361, WKN 822362, WKN 822380, and WKN 822399 which are traded at the stock exchanges in Frankfurt, Düsseldorf, and Stuttgart.

The data, again, are taken from Rubinstein's example with an additional cap of 10. Again, this PDE can be solved either on a square or triangular domain. For reasons outlined above, we chose the triangle. The boundary conditions at $S_1 = 0$ and $S_2 = 0$ represent the prices of capped European call options with strike prices of $\frac{X}{w_2}$ and $\frac{X}{w_1}$, respectively. This capped call can be priced either by entering a bull spread and pricing its parts individually with the analytical formula for European calls or numerically.

Volatility		Time to Maturity		
σ_1^2	σ_2^2	0.05	0.5	0.95
0.1	0.1	1.9062	5.3075	6.1865
	0.2	2.5549	4.9940	5.4814
	0.3	3.0139	4.7564	5.0111
0.2	0.1	2.5534	5.0002	5.5096
	0.2	2.9701	4.8713	5.1819
	0.3	3.2831	4.7214	4.8849
0.3	0.1	3.0124	4.7643	5.0459
	0.2	3.2825	4.7260	4.9005
	0.3	3.4980	4.6436	4.7225

Table 25: Results Capped Call on a Basket with Analytical Boundary Conditions

Again, we do not have a direct way of checking the results. By arbitrage considerations, however, each premium should be worth less than the example above with a rebate of $R = 10$ table (24) and more than the example without rebate table (21).

Gradient Boundary Conditions An alternative to eq. (56 and (57) is to assume a zero gradient on $S_1 = 0$ and $S_2 = 0$. The results differ only slightly; compare table (26).

Volatility		Time to Maturity		
σ_1^2	σ_2^2	0.05	0.5	0.95
0.1	0.1	1.9073	5.3112	6.1885
	0.2	2.5604	4.9959	5.4823
	0.3	3.0206	4.7575	5.0116
0.2	0.1	2.5559	5.0002	5.5110
	0.2	2.9735	4.8723	5.1825
	0.3	3.2873	4.7220	4.8853
0.3	0.1	3.0138	4.7654	5.0467
	0.2	3.2844	4.7266	4.9011
	0.3	3.5009	4.6442	4.7223

Table 26: Results Capped Call on a Basket with Numerical Boundary Conditions

3.3.6 Capped Power Call on a Basket with a Down-and-out Barrier

In this section we apply some of the exotic features of the previous sections on a symmetric power call on a basket. The power parameter is set to $p = 2$. Additionally, we introduce a cap of 20 on the option and a down-and-out barrier when the basket becomes worth less than 100.

Volatility		Time to Maturity		
σ_1^2	σ_2^2	0.05	0.5	0.95
0.1	0.1	5.5636	11.8290	13.1566
	0.2	6.8347	10.7868	11.5071
	0.3	7.5183	10.1014	10.4334
0.2	0.1	6.8742	10.8144	11.5788
	0.2	7.5100	10.3587	10.7864
	0.3	7.9196	9.9305	10.1068
0.3	0.1	7.5528	10.1310	10.5182
	0.2	7.9261	9.9433	10.1431
	0.3	8.1958	9.6955	9.7287

Table 27: Results Power Call on a Basket with Floor and Knock-out Barrier

4 Conclusions

In the previous sections it has been demonstrated how to use FE to price options of various kinds. It has been delineated that the FEM has some advantages in computing accurate Greeks due to its polynominal approximation of the PDE. It has also been outlined how non-rectangular domains arise in option pricing and how to deal with these in a FE setting. This has been demonstrated with various options on baskets, but this can easily be generalized to other rainbow options. The possibility of being able to handle arbitrary domains is the main reason for the predominance of FE in civil and mechanical engineering. This allows a wealth of knowledge and software to be on hand. The package used for this paper is *PDE*ase*2D*^{TM}, clearly its high accuracy has been demonstrated. A computer run for a single problem takes from a few seconds to several minutes.[13] Since *PDE*ase*2D*^{TM} is a general purpose program for linear and nonlinear PDEs of various types and arbitrary domains, the solution process could be made substantially faster by coding only parabolic PDEs.

[13]Since many different PCs were used for this paper, CPU time of individual problems are not shown.

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